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A NOTE ON THE FLOW OF A FLUID WITH PRESSURE-DEPENDENT VISCOSITY THROUGH A POROUS MEDIUM WITH VARIABLE PERMEABILITY

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Abstract. Flow of a fluid with pressure-dependent viscosity is considered through a porous channel with variable permeability in order to illustrate the determinate nature of the governing equations and to study the effect of fluid and domain parameters on the flow characteristics.

Keywords: *pressure-dependent viscosity, Brinkman equation, variable permeability, exact solution.*

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1. Introduction

Interest in flow of fluids with pressure-dependent viscosities (also referred to as pressure-thickening [12]) goes back to the nineteenth century and the work of Barus, [4,5], who suggested a relationship in which viscosity is an exponential function of pressure. Since then, the subject matter has been receiving considerable attention in the literature due to its various applications in industry, including lubrication theory (cf. [6] and the references therein). A number of models describing the relationship between viscosity and pressure in an incompressible fluid have been suggested and tested, and include exponential as well as linear relationships, (cf. [3,13,21] and the references therein). A general model describing the dependence of viscosity on pressure, temperature and density has been reported in Szeri [21]. In case of compressible flow, Housiadas and Georgiou [9] provided new solutions for weakly compressible Newtonian Poiseuille flows with pressure-dependent viscosity.

An important problem of the flow of fluids with pressure-dependent viscosities is that of flow through porous media. This finds applications in various industrial and natural processes including groundwater and oil recovery (cf. [8,14] and the references therein). Various models and elegant analyses have been provided in the pioneering work of Rajagopal and coworkers (cf. [11, 15, 16, 17, 19, 20] and the references therein). Modelling of flow of dusty fluids with pressure-dependent viscosities through porous structures has also been considered due to its applications in contaminant transport into ground layers, [1, 7].

Less studied, however, is the flow of fluids with pressure-dependent viscosities through porous structures with variable porosity and permeability, which is more reflective of flow through natural structures, [7]. This is the subject matter of this work in which we study flow in an inclined porous channel of variable permeability using a model that has been developed to account for permeability variations [2]. The physical configuration is suitable for studying characteristics of the flow and has been used by other authors in the analysis of fluid flow with variable viscosity, [11, 18]. For the sake of illustration, we consider variations in permeability that are the square of pressure distribution.

2. Problem formulation and solution

Consider the flow of a fluid with pressure-dependent viscosity through a porous sediment of variable permeability, and of depth *h* inclined at angle ϑ to the horizontal. The flow configuration is illustrated in Fig. 1 which shows the orientation of the coordinate system used. It is assumed that the sediment is bounded by impermeable, solid walls on which the no-slip condition holds.

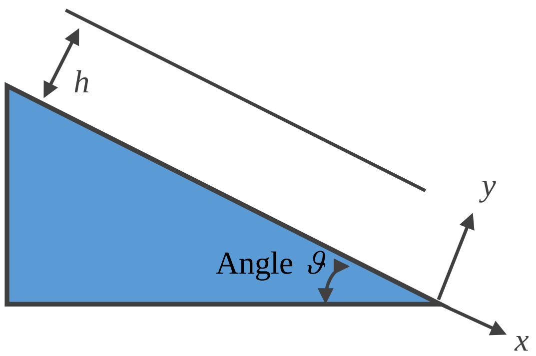

Fig. 1. Representative sketch

Flow in the above domain is governed by the equation of continuity and momentum equations, developed in [2] and are given, respectively, by:

Continuity Equation:

$$\nabla \bullet \vec{u} = 0 \tag{1}$$

Momentum Equations:

$$\rho\{\vec{u}_t + \vec{u} \bullet \nabla \vec{u}\} = -\nabla p + \mu \nabla \bullet (\nabla \vec{u} + (\nabla \vec{u})^T) + (\nabla \mu) \bullet (\nabla \vec{u} + (\nabla \vec{u})^T)$$
$$-\frac{\mu}{k}\vec{u} + \rho \vec{G} \tag{2}$$

where $\vec{u}$ is the velocity vector, P is the pressure, ρ is the fluid density, $\vec{G}$ is the gravitational acceleration, $\mu = \mu(p)$ is the variable viscosity, and *k* is the variable permeability.

Equations (1) and (2) reduce to the following set of equations when the flow is through the configuration of Fig. 1:

$$-p_x + \mu u'' + \mu' u' + \rho g \sin\vartheta - \frac{\mu}{k} u = 0 \tag{3}$$

$$-p_y - \rho g \cos\vartheta = 0 \tag{4}$$

with boundary conditions given by zero-slip on the solid boundaries y=0 and y=h, and a prescribed pressure at y=h (such as atmospheric pressure, say p_0). Boundary conditions are thus given as:

$$\left.\begin{aligned} u(0) &= 0 \\ u(h) &= 0 \\ p(h) &= p_0 \end{aligned}\right\} \tag{5}$$

Following Kanaan and Rajagopal, [11], assume that $p = p(y)$ and introduce the dimensionless quantities:

$$\bar{y} = y/h, \bar{u} = u/U \tag{6}$$

then boundary conditions (5) take the form

$$\left.\begin{aligned} \bar{u}(0) &= 0 \\ \bar{u}(1) &= 0 \\ p(1) &= p_0 \end{aligned}\right\} \tag{7}$$

and governing equations (3) and (4) can be written, respectively, as

$$\mu \frac{d^2\bar{u}}{d\bar{y}^2} + \frac{d\mu}{d\bar{y}}\frac{d\bar{u}}{d\bar{y}} + \frac{\rho g h^2}{U}\sin\vartheta - \frac{h^2\mu}{k}\bar{u} = 0 \tag{8}$$

$$\frac{dp}{d\bar{y}} = -\rho g h \cos\vartheta . \tag{9}$$

General solution to (9) takes the form

$$p = (-\rho g h \cos\vartheta)\bar{y} + c \tag{10}$$

where c is an arbitrary constant.

Using pressure condition $p(1) = p_0$, we find that $c = p_0 + \rho g h \cos\vartheta$, hence (10) takes the form

$$p = p_0 + (1-\bar{y})\rho g h \cos\vartheta . \tag{11}$$

In order to solve (8) for $\bar{u}(\bar{y})$, we assume that the viscosity varies with pressure according to

$$\mu(p) = \alpha p \tag{12}$$

where α is a constant.

From (12) we obtain

$$\frac{d\mu}{d\bar{y}} = \frac{d\mu}{dp}\frac{dp}{d\bar{y}} = \alpha\frac{dp}{d\bar{y}} \tag{13}$$

and (11) gives

$$\frac{dp}{d\bar{y}} = -\rho g h \cos\vartheta . \tag{14}$$

Equation (8) thus becomes

$$p^2\frac{d^2\bar{u}}{d\bar{y}^2}-[\rho gh\cos\vartheta]p\frac{d\bar{u}}{d\bar{y}}-\frac{h^2p^2}{\bar{k}}\bar{u}=-p\frac{\rho gh^2}{\alpha U}\sin\vartheta\,. \tag{15}$$

In order to solve (15), we need to specify the permeability distribution. In the current work, we choose permeability $k=p^2$ for the sake of illustration, and assume this choice is valid for $0<\bar{y}<1$ while permeability falls to zero at the solid boundaries. With this choice of permeability equation (15) takes the form

$$p^2\frac{d^2\bar{u}}{d\bar{y}^2}-[\rho gh\cos\vartheta]p\frac{d\bar{u}}{d\bar{y}}-h^2\bar{u}=-p\frac{\rho gh^2}{\alpha U}\sin\vartheta \tag{16}$$

Which, when (11) is utilized, becomes

$$\begin{aligned}&[p_0+(\rho gh\cos\vartheta)(1-\bar{y})]^2\frac{d^2\bar{u}}{d\bar{y}^2}-[\rho gh\cos\vartheta][p_0+(\rho gh\cos\vartheta)(1-\bar{y})]\frac{d\bar{u}}{d\bar{y}}\\&-h^2\bar{u}=-[p_0+(\rho gh\cos\vartheta)(1-\bar{y})]\frac{\rho gh^2}{\alpha U}\sin\vartheta\end{aligned} \tag{17}$$

Letting

$$Y=p_0+(\rho gh\cos\vartheta)(1-\bar{y}) \tag{18}$$

we can write equation (17) in the form:

$$Y^2\frac{d^2\bar{u}}{dY^2}+Y\frac{d\bar{u}}{dY}-\frac{1}{[\rho g\cos\vartheta]^2}\bar{u}=-Y\frac{\sin\vartheta}{\alpha U\rho g\cos^2\vartheta}\,. \tag{19}$$

Equation (19) is an inhomogeneous Cauchy-Euler equation whose complementary solution is given by

$$\bar{u}_c=a_1Y^{m_1}+a_2Y^{m_2} \tag{20}$$

where a_1 and a_2 are arbitrary constants, and

$$m_1=\frac{1}{\rho g\cos\vartheta} \tag{21}$$

and

$$m_2=-\frac{1}{\rho g\cos\vartheta}\,. \tag{22}$$

The characteristic roots, m_1 and m_2 satisfy the following equations

$$m_2-m_1=-\frac{2}{[\rho g\cos\vartheta]} \tag{23}$$

$$m_1+m_2=0 \tag{24}$$

and the non-zero Wronskian of the solutions in (20) is given by

$$W(Y^{m_1},Y^{m_2})=-\frac{2}{[\rho g\cos\vartheta]Y}\,. \tag{25}$$

Using variation of parameters, the particular solution to (19) is constructed and takes the form

$$\bar{u}_p = -\frac{\sin\vartheta}{\alpha U}\frac{\rho g Y}{[\rho g\cos\vartheta]^2 - 1} = -\frac{\sin\vartheta}{\alpha U}\frac{\rho g\left[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})\right]}{[\rho g\cos\vartheta]^2 - 1}. \tag{26}$$

General solution to (19) is thus the sum of solutions in (20) and (26), namely

$$\bar{u} = a_1 Y^{m_1} + a_2 Y^{m_2} - \frac{\sin\vartheta}{\alpha U}\frac{\rho g[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})]}{[\rho g\cos\vartheta]^2 - 1}. \tag{27}$$

Vorticity of the flow is obtained from (27) as

$$\begin{aligned}\bar{\omega} = -\bar{u}_{\bar{y}} &= a_1 h\frac{\left[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})\right]^{\left(\frac{1}{\rho g\cos\vartheta}\right)}}{\left[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})\right]} \\ &- a_2 h\frac{\left[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})\right]^{\left(-\frac{1}{\rho g\cos\vartheta}\right)}}{\left[p_0 + (\rho g h\cos\vartheta)(1-\bar{y})\right]} - \frac{\rho^2 g^2 h\sin\vartheta}{2\alpha U(\rho^2 g^2\cos^2\vartheta - 1)}.\end{aligned} \tag{28}$$

The arbitrary constants a_1 and a_2 take the following values, obtained by using conditions (7) in (27):

$$a_1 = -\frac{\rho g\sin\vartheta}{\alpha U([\rho g\cos\vartheta]^2 - 1)}\frac{p_0\left(p_0 + (\rho g h\cos\vartheta)\right)^{m_2} - \left(p_0 + (\rho g h\cos\vartheta)\right)\left(p_0\right)^{m_2}}{\left(p_0 + (\rho g h\cos\vartheta)\right)^{m_1}\left(p_0\right)^{m_2} - \left(p_0 + (\rho g h\cos\vartheta)\right)^{m_2}\left(p_0\right)^{m_1}} \tag{29}$$

$$a_2 = -\frac{\rho g\sin\vartheta}{\alpha U([\rho g\cos\vartheta]^2 - 1)}\frac{-p_0\left(p_0 + (\rho g h\cos\vartheta)\right)^{m_1} + \left(p_0\right)^{m_1}\left(p_0 + (\rho g h\cos\vartheta)\right)}{\left(p_0 + (\rho g h\cos\vartheta)\right)^{m_1}\left(p_0\right)^{m_2} - \left(p_0 + (\rho g h\cos\vartheta)\right)^{m_2}\left(p_0\right)^{m_1}}. \tag{30}$$

3. Results and discussion

Velocity and vorticity profiles across the channel are obtained for the following ranges of flow and domain parameters: h=1, $\rho g = 1$, $\alpha U = 0.5, 0.8, 1, 2, 5, 10, 20, 50$, $\vartheta = 30, 60, 75$ degrees, and $p_0 = 2, 3, 5$.

3.1. Values of the Characteristic Roots and Arbitrary Constants:

Values of the characteristic roots are computed using equations (21) and (22). For $\vartheta = 30^\circ$, $\rho g = 1$ the computed values are $m_1 = 1.154700539$ and $m_1 = -1.154700539$. The corresponding values of arbitrary constants a_1 and a_2 for various values of αU, are computed using expressions (29) and (30 and tabulated in Table 1, for the sake of illustration.

Table 1: Values of a_1, a_2 for $h = 1$, $p_0 = 2$, $\rho g = 1$, $\vartheta = 30$

αU	a_1	a_2
0.5	-3.248565346	-1.708648731
0.8	-2.030353341	-1.067905457
1	-1.624282673	-0.8543243654
2	-0.8121413365	-0.4271621827
5	-0.3248565346	-0.1708648731
10	-0.1624282673	-0.08543243654
20	-0.08121413365	-0.04271621827
50	- 0.03248565346	- 0.01708648731

3.2. Pressure and Permeability Distributions

Pressure and permeability distributions across the channel for different values of p_0 are shown in Fig. 2(a) and 2(b) in order to illustrate their dependence on p_0. Effects of varying ϑ on these distributions are shown in Fig. 3(a) and 3(b) to illustrate the decrease in pressure and in permeability with increasing angle of inclination. These figures are obtained by sketching equation (11) and $k = p^2$. The permeability distribution in Fig. 2(b) and 3(b) are over the interval $0 < \bar{y} < 1$, since we assumed that the permeability vanishes on the solid walls.

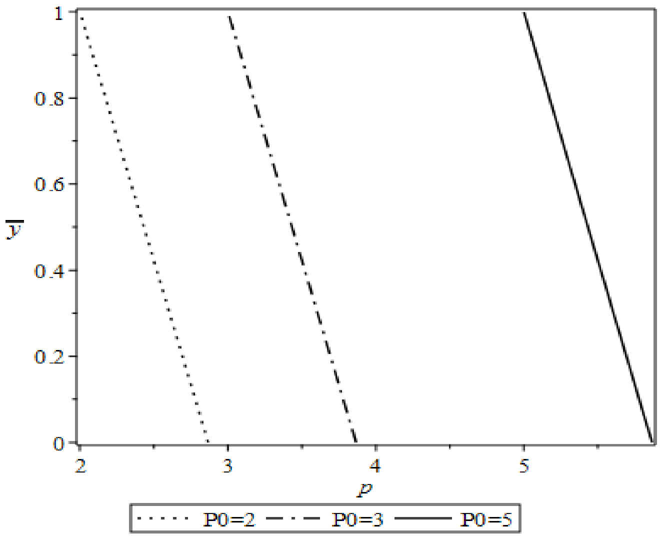

Fig. 2(a). Pressure distribution across the channel for different p_0;

$\vartheta = \pi/6$, $\rho g = 1$, h=1.

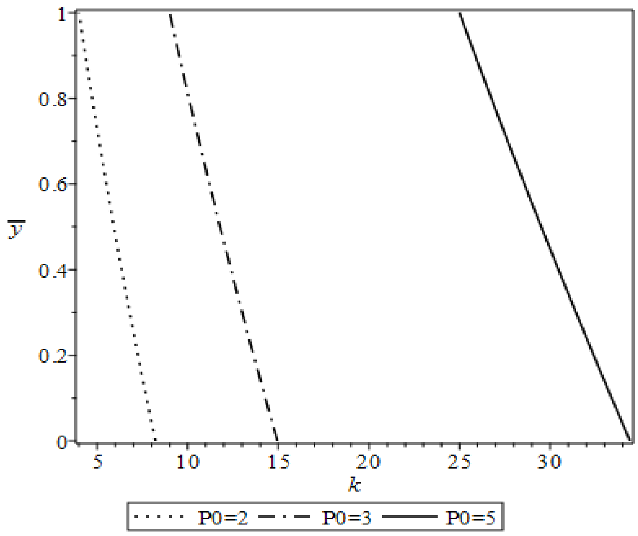

Fig. 2(b). Permeability distribution over $0 < \bar{y} < 1$ for different p_0;

$\vartheta = 30^{\circ}$, $\rho g = 1$, h=1.

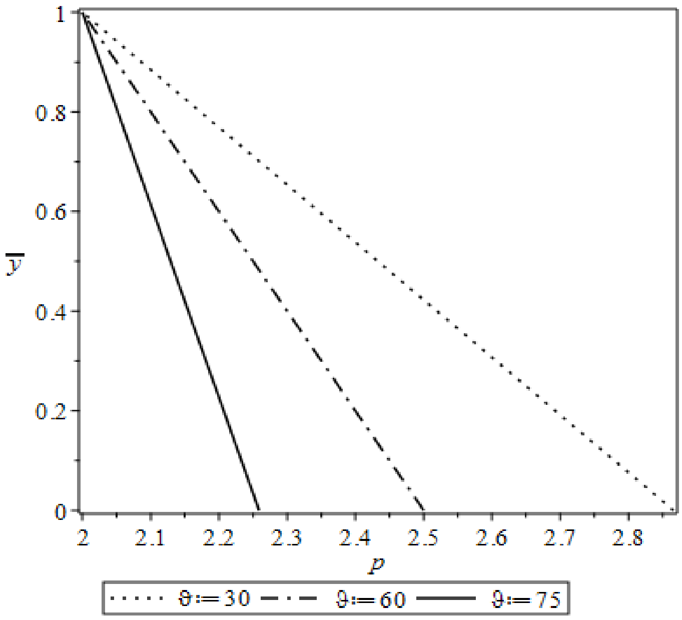

Fig. 3(a). Pressure distribution across the channel for different ϑ;
$p_0 = 2$, $\rho g = 1$, h=1.

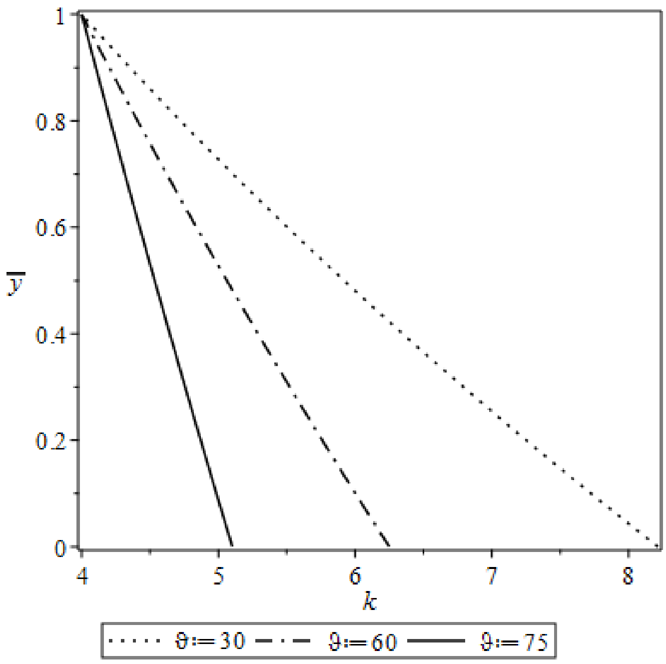

Fig. 3(b). Permeability distribution over $0 < \bar{y} < 1$ for different ϑ;
$p_0 = 2$; $\rho g = 1$, h=1.

3.3. Effects of Varying αU

Effects of varying αU on the velocity profile for a fixed angle of inclination and a fixed p_0 are illustrated in Fig. 4(a), which depicts a plot of the parabolic velocity profile in (27) and shows a decrease in the velocity with increasing αU. This may be interpreted in terms of (12), where viscosity is a linear function of pressure magnified by a factor of α. When α increases (hence αU increases), the fluid becomes more viscous and slower. This decrease in velocity with increasing αU can also be seen from (27), as this quantity appears in the denominator of the particular solution, hence its increase results in decreasing the contribution of the particular solution.

Effects of varying αU on the vorticity profile for a fixed angle of inclination and a fixed p_0 are illustrated in Fig. 4(b), which depicts a plot of the vorticity in (28), and shows an increase in the slope of the tangent to vorticity curves with increasing αU. This may be interpreted in terms of the definition of vorticity as $\bar{\varpi} = -\bar{u}_{\bar{y}}$ and the decrease of velocity with increasing αU, thus implying that the parabolic velocity profile flattens or loses some of its curvature. In other words, the rate of change of the velocity with respect to the independent variable decreases in magnitude. For a given viscosity distribution, this also implies that the magnitude of shear stress decreases with increasing αU.

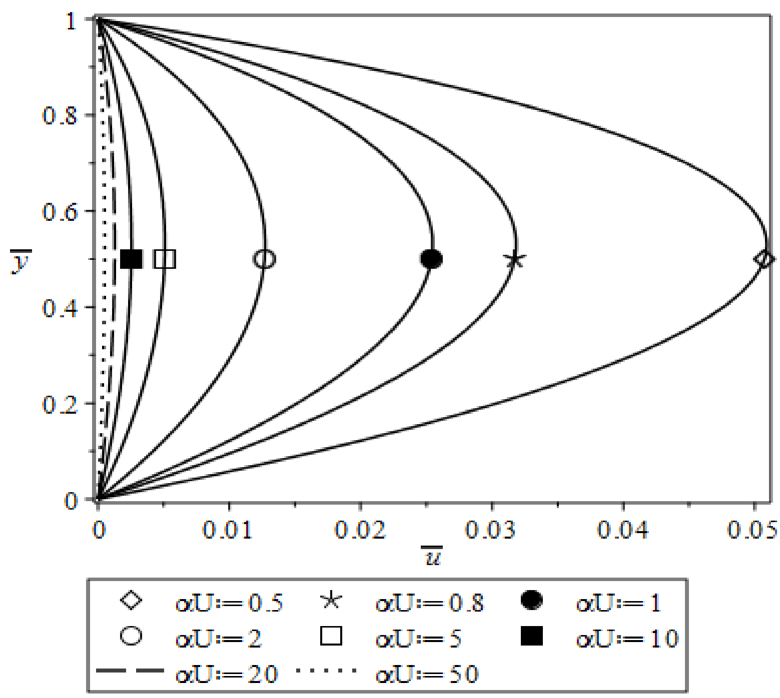

Fig. 4(a). Effects of αU on the Velocity Profile;
$h = 1$, $p_0 = 2$, $\rho g = 1$, $\vartheta = 30$

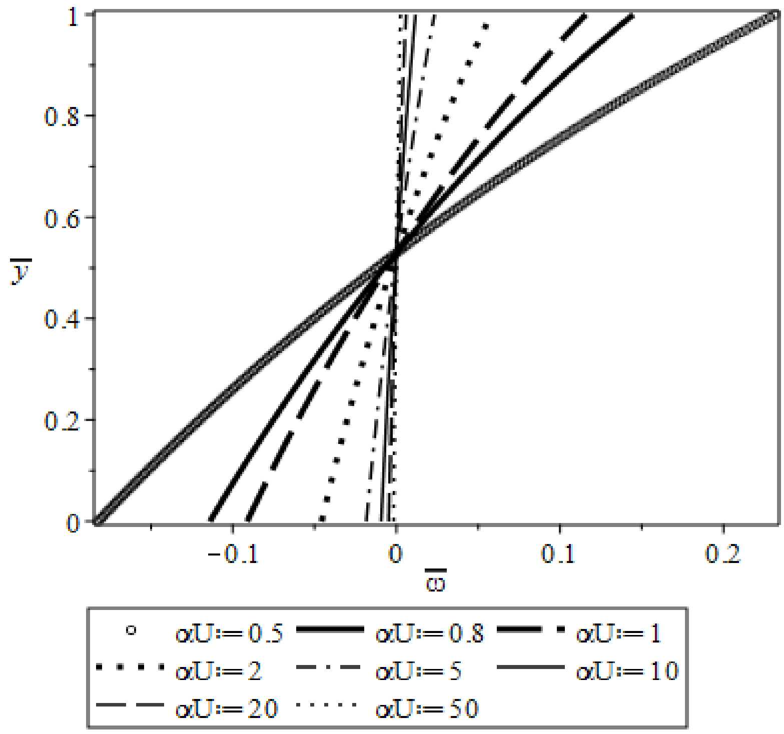

Fig. 4(b). Effects of αU on the Vorticity Profile;
$h=1,\ p_0=2,\ \rho g=1,\ \vartheta=30$

3.4. Effects of Varying p_0

Effects of varying p_0 on the velocity profile are illustrated in Fig. 5(a) and 5(b), respectively, for the choice of parameters: $h=1,\ \alpha U=1,\ \rho g=1,\ \vartheta=30^\circ$. For a fixed value of αU, hence a fixed value of α, equation (12) implies that viscosity increases with increasing pressure. Equation (11) indicates that pressure increases with increasing p_0 (for the fixed parameters chosen here). This in turn implies that viscosity increases with increasing p_0, thus decreasing the velocity. This is demonstrated in Fig. 5(a) where the velocity decreases with increasing p_0.

The decrease in velocity with increasing p_0 results in flattening the velocity profile, which in turn results in a decrease in the magnitude of the vorticity. This is demonstrated in Fig. 5(b).

3.5. Effects of Varying ϑ

Effects of varying the angle of inclination, ϑ, on the velocity and vorticity distributions across the channel are illustrated in Fig. 6(a) and 6(b). With increasing ϑ, gravitational effects become more pronounced, thus leading to faster flow down the inclined plane, as shown in Fig. 6(a), and a corresponding increase in the magnitude of vorticity, as shown in Fig. 6(b).

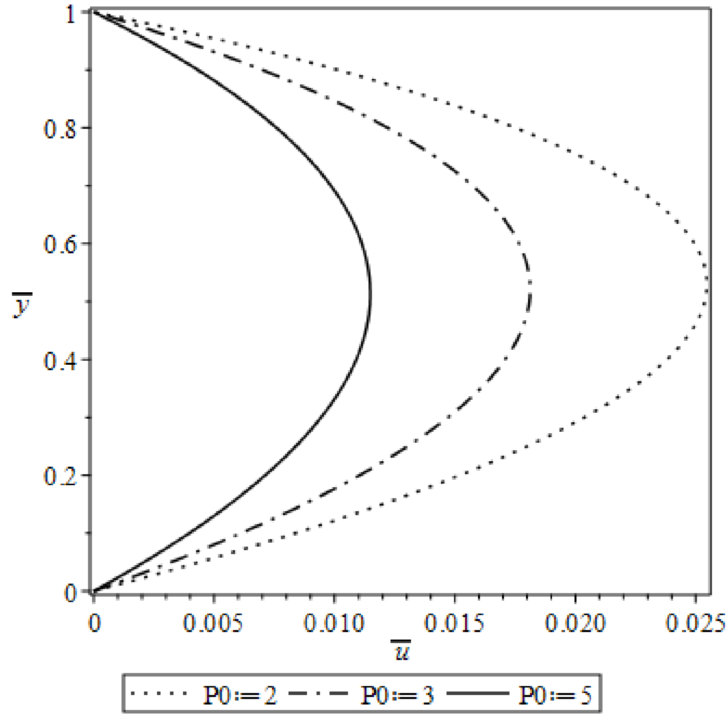

Fig. 5(a). Effects of Varying p_0 on Velocity Profile;

$h = 1,\ \alpha U = 1,\ \rho g = 1,\ \vartheta = 30^{\circ}.$

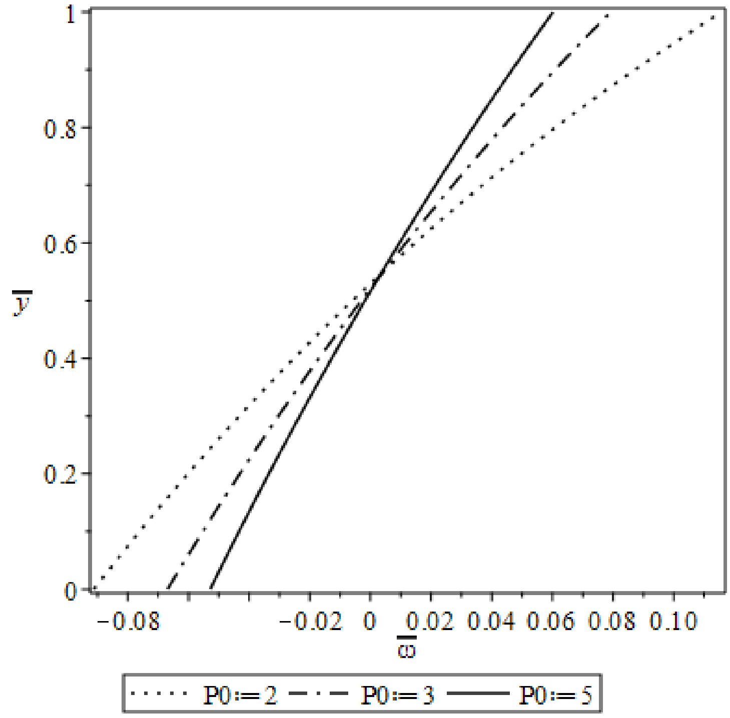

Fig. 5(b). Effects of Varying p_0 on Velocity Profile;

$h = 1,\ \alpha U = 1,\ \rho g = 1,\ \vartheta = 30^{\circ}.$

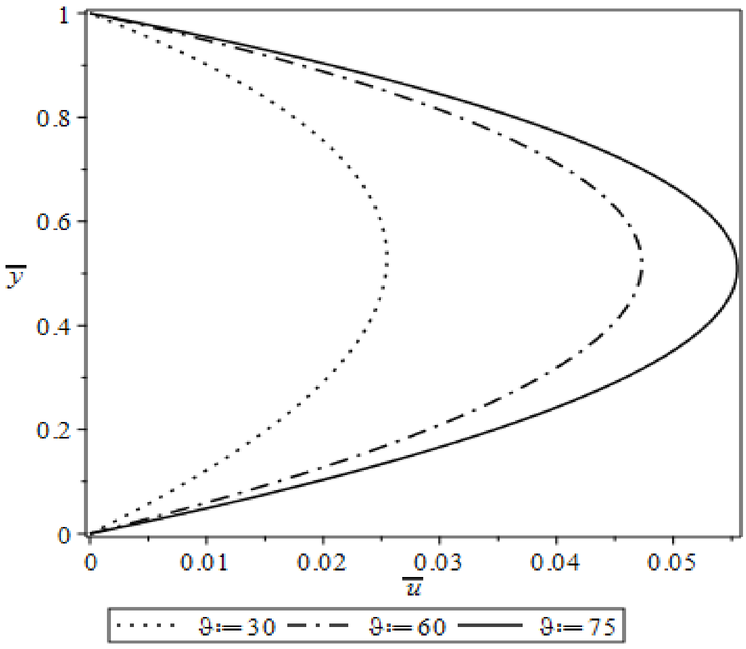

Fig. 6(a). Effects of varying on the velocity distribution across the channel.

$$h = 1,\ p_0 = 2,\ \alpha U = 1,\ \rho g = 1$$

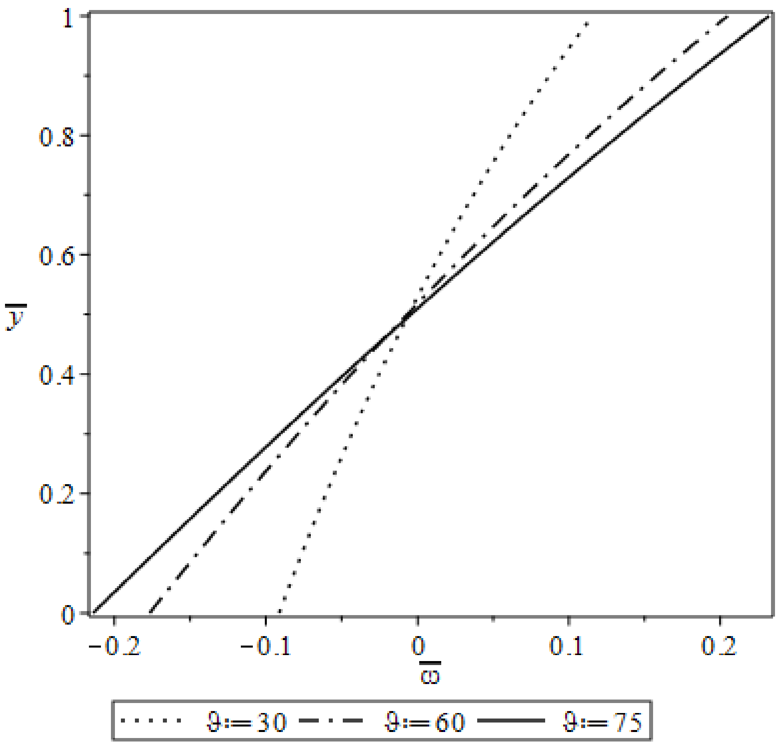

Fig. 6(b). Effects of varying ϑ on the vorticity distribution across the channel;

$$h = 1,\ p_0 = 2,\ \alpha U = 1,\ \rho g = 1$$

4. Conclusion

In this work, we obtained the solution to flow of a fluid with pressure dependent viscosity through a porous medium with variable permeability. The configuration chosen was that of flow down an inclined channel to illustrate the determinate nature of a recently developed model. The variable permeability was chosen to be the square of pressure for illustrative purposes. However, other choices of variable permeability distributions are possible and depend on the application sought.

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